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New non-linear evolution equations related to some superloop algebras

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Abstract. We construct hierarchies of non-linear evolution equations on the superloop algebras related to the super Lie algebras b(0, 1) and sl(2/1). The hierarchy associated with sl(2/1) contains a generalisation of the super-non-linear Schrödinger equation, with a cubic, non-derivative interaction term, in complex anticommuting fields. In all cases we give a recursive procedure for constructing involutive integrals of local polynomials.

1. Introduction

Supersymmetry has become an important and fruitful concept in theoretical physics [1]. Its importance became clear to particle physicists with the fundamental work of Wess and Zumino [2]. Since then interest in supersymmetry has grown rapidly, and in addition to particle physics and field theory, several applications have been attempted in such diverse fields as nuclear physics, solid state physics and statistical mechanics. For general review articles on these subjects see, for example, [3].

In the late 1970s Kac [4] worked out a complete classification of simple super-Lie algebras. He proved that they could be described by a Cartan matrix, i.e. by a generalisation of Dynkin diagrams, now called Kac-Dynkin diagrams. Since then several important contributions have been made, both to the representation theory of super-Lie algebras and to their further application in theoretical physics.

One area of particular interest is integrable super-non-linear evolution equations (super-NLEE). In recent years several authors have extended integrable classes of NLEE to include anticommuting field variables. Chaichian and Kulish [5] have studied the super-Liouville and super-sine-Gordon equations and shown that the formalism of the inverse problem can be applied to these equations. They found that these equations correspond to the super-Lie algebras osp(2, 1) and sl(2/1) respectively. Kupershmidt [6] has constructed a super-extension of the KdV and the MKdV equation, and proved them to be completely integrable.

Most super-NLEE have been considered individually, rather than as members of a hierarchy. Gürses and Oğuz [7] gave a generalisation of the classical AKNS problem to a hierarchy based on the super-Lie algebra b(0, 1). It contains a super-extension of the NLS equation [8] and the super-KdV and super-MKdV equations considered by Kupershmidt [6]. The complete integrability of the super-NLS equation has been proved by Chowdhury and Naskar [9]. Gürses and Oğuz did not discuss the Hamiltonian structure of the equations. This was done by Chowdhury and Roy [10] by using the technique of Riccati equations and an extension of the variational approach of Tu [11].

In a previous paper [12] a general method was developed to construct a hierarchy of Hamiltonian NLEE on an arbitrary loop algebra. In this paper some of the results in [12] are extended to include super-Lie algebras. A system of super-NLEE is constructed, and their infinite non-trivial set of conserved quantities is found. Applications to the superloop algebras associated with b(0, 1) and sl(2/1) are considered.

We find that the equations related to b(0, 1) include a super-version of the NLS equation [7, 8] together with the super-KdV and super-MKdV equation [6, 7]. The equations related to sl(2/1) contain a generalisation of the super-NLS equation with a cubic non-derivative interaction in complex anticommuting fields. A term of this kind can only be present if the root space of the algebra under consideration contains more than one odd root, but each root vector associated with an odd root of a given sign gives rise to one anticommuting field. The root space for sl(2/1) contains two odd roots and their root vectors give rise to four anticommuting fields. This explains the presence of the cubic spinor term in the generalised super-NLS equation associated with sl(2/1).

2. Formulation

Let $Z_2 = \{0, 1\}$ be the group of integers under addition modulo 2. The super-Lie algebra $g = g_0 + g_1$ is a Z_2-graded Lie algebra with a super-Lie bracket [,] which satisfies [4]

$$[X, Y] = XY - (-1)^{d(X)d(Y)} YX \tag{2.1}$$

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{d(X)d(Y)}[Y, [X, Z]] \tag{2.2}$$

where $d(X) = 0\ (1)$ if $X \in g_0\ (g_1)$; $d(X)$ is called the degree of X. Hereafter we write $[\ ,\ ]_-$ for commutators and $[\ ,\ ]_+$ for anticommutators.

Let h be a Cartan subalgebra of g_0 and h^* the dual of h. For $\alpha \in h^*$, $\alpha \neq 0$

$$g_\alpha = \{X \in g: [H, X] = \alpha(H)X, H \in h\} \tag{2.3}$$

is called the root space and α is a root if $g_\alpha \neq 0$. Let Δ be the set of roots. $\alpha \in \Delta$ is said to be even, respectively odd, if $g_\alpha \cap g_0 \neq 0$, respectively $g_\alpha \cap g_1 \neq 0$. Even and odd roots are denoted by Δ_0 and Δ_1 respectively. On g there is defined a non-degenerate bilinear form $\langle\ ,\ \rangle$ which has the following properties: it is consistent, $\langle X, Y\rangle = 0$ for $X \in g_0$ and $Y \in g_1$, it is supersymmetric, $\langle X, Y\rangle = (-1)^{d(X)d(Y)}\langle Y, X\rangle$, and it is invariant $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$.

Let $\hat{g} = \Sigma_{j\in z}\, g^{(j)}$, $X^{(j)} = X_j\lambda^j \in g^{(j)}$, $X_j \in g$, be a loop algebra over g and $g_0^+ = \Sigma_{j\geq 0}\, g^{(j)}$, $g^- = \Sigma_{j\leq -1}\, g^{(j)}$. $\hat{g}$ is equipped with a non-degenerate bilinear form $(X, Y) = \mathrm{Res}(\langle X, Y\rangle\lambda^{-1})$. This form is consistent, supersymmetric and invariant. Any function on $\hat{g}$, or a subspace thereof, has a polynomial form $f(X) = \Sigma_j f(X_j)\lambda^j, f(X) \in C$ and is therefore a well defined element in $C[\lambda, 1/\lambda]$, the Laurent polynomials in λ. One can identify $(g_0^+)^*$, the dual of g_0^+, with $g_0^- = \Sigma_{j\leq 0}\, g^{(j)}$ [12, 13] and it is equipped with the natural super-Poisson bracket:

$$\{f, g\}_\pm = (X, [P\nabla f(X), P\nabla g(X)]_\pm]) \qquad f, g \in C((g_0^+)^*) \tag{2.4}$$

where P is the projection $\hat{g} \to g_0^+$.

Let $A = A((g_0^+)^*)$ be a ring of ad-invariant polynomials on $(g_0^+)^*$. The theorem of Adler, Kostant and Symes [14], which can be extended to super-Lie algebras, tells us that every $\varphi \in A$ gives rise to a vector field on $(g_0^+)^*$ given by

$$\Delta_\varphi(X) = [\Pi \nabla \varphi(X), X] \tag{2.5}$$

where $\Pi: \hat{g} \to g^-$. The functions on $(g_0^+)^*$:

$$\rho_k(X) = -\tfrac{1}{2} \operatorname{Res}(\lambda^{-k-1} \langle X, X \rangle) \qquad k \in Z_+ \tag{2.6}$$

are in involution with respect to the super-Poisson bracket (2.4). For every k they give rise to a vector field of the form [12]

$$\Delta_k(X) = -\sum_{j \geq k+1} [X_j \lambda^{k-j}, X]. \tag{2.7}$$

After equating different powers of λ we obtain

$$X_{0;t_k} = 0 \tag{2.8a}$$

$$X_{n;t_k} = \sum_{j=0}^{\min(n-1,\,k)} [X_j, X_{k+n-j}] \tag{2.8b}$$

where $;t_k$ is the derivative with respect to the parameter of the vector field Δ_k. The bracket [,] is to be considered as a commutator or an anticommutator, depending on the degree of the generators in it. Generally, $X_j = X_j^0 + X_j^1$, $X_j^0 \in g_0$, $X_j^1 \in g_1$. Then (2.8) becomes

$$X^0_{0;t_k} = 0 \qquad X^1_{0;t_k} = 0 \tag{2.9a}$$

$$X^0_{n;t_k} = \sum_{j=0}^{\min(n-1,k)} ([X_j^0, X^0_{k+n-j}]_- + [X_j^1, X^1_{k+n-j}]_+) \tag{2.9b}$$

$$X^1_{n;t_k} = \sum_{j=0}^{\min(k,\,n-1)} ([X_j^1, X^0_{k+n-j}]_- + [X_j^0, X^1_{k+n-j}]_-). \tag{2.9c}$$

We consider X_j as a function with values in g. Let B_a, F_α with $a = 1, \ldots, m$, $\alpha = 1, \ldots, n$ be some basis in g satisfying

$$[B_a, B_b]_- = C^c_{ab} B_c \qquad [B_a, F_\alpha]_- = C^\beta_{a\alpha} F_\beta \qquad [F_\alpha, F_\beta]_+ = C^a_{\alpha\beta} B_a. \tag{2.10}$$

Except when otherwise mentioned, summation in repeated indices is understood. In this basis we set

$$X_j = q_j^a B_a + \psi_j^\alpha F_\alpha \tag{2.11}$$

where q_j^a and ψ_j^α are fields that depend on the infinite set of parameters $t_1, t_2, t_3, \ldots$. The fields ψ_j^α anticommute with each other and commute with b_j^a. The b_j^a commute among themselves:

$$\psi_i^\alpha \psi_j^\beta + \psi_j^\beta \psi_i^\alpha = 0 \qquad \psi_i^\alpha b_j^a - b_j^a \psi_i^\alpha = 0 \qquad b_i^a b_j^c - b_j^c b_i^a = 0. \tag{2.12}$$

Inserting (2.11) into (2.9) we find the following equations for the fields $q_j^a(t_1, t_2, \ldots)$

and $\psi_j^\alpha(t_1, t_2, \ldots)$:

$$q^a_{0;t_k} = 0 \qquad \psi^\alpha_{0;t_k} = 0 \tag{2.13a}$$

$$q^a_{n;t_k} = \sum_{j=0}^{(n-1,k)} (C^a_{bc} q^b_j q^c_{k+n-j} - C^a_{\beta\gamma} \psi^\beta_j \psi^\gamma_{k+n-j}) \tag{2.13b}$$

$$\psi^\alpha_{n;t_k} = \sum_{j=0}^{(n-1,k)} C^\alpha_{a\beta} (q^a_j \psi^\beta_{k+n-j} - \psi^\beta_j q^a_{k+n-j}) \tag{2.13c}$$

with $(n-1, k) = \min(n-1, k)$. In a suitable basis for g these equations give rise to super-NLEE in a recursive form.

3. Examples

3.1. $g = b(0, 1)$

This super-Lie algebra has five generators, three even (H, E, F) and two odd (Q, R), satisfying the following (anti)commutation relations:

$$\begin{array}{lll} [H, E]_- = 2E & [H, F]_- = -2F & [E, F]_- = H \\ [H, Q]_- = Q & [H, R]_- = -R & \\ [E, R]_- = Q & [F, Q]_- = R & [Q, Q]_+ = -2E \\ [R, R]_+ = 2F & [Q, R]_+ = H. & \end{array} \tag{3.1}$$

Expand X_j in this basis:

$$X_j = h_j H + e_j E + f_j F + q_j Q + r_j R \tag{3.2}$$

where the functions h_j, e_j, f_j commute and the functions q_j and r_j anticommute. From (2.9) we find the following equations:

$$h_{0;t_k} = e_{0;t_k} = f_{0;t_k} = 0 \qquad q_{0;t_k} = r_{0;t_k} = 0 \tag{3.3a}$$

$$h_{n;t_k} = \sum_{j=0}^{(k,n-1)} (e_j f_{k+n-j} - f_j e_{k+n-j} + r_j q_{k+n-j} + q_j r_{k+n-j}) \tag{3.3b}$$

$$e_{n;t_k} = \sum_{j=0}^{(k,n-1)} 2(h_j e_{k+n-j} - e_j h_{k+n-j} + q_j q_{k+n-j}) \tag{3.3c}$$

$$f_{n;t_k} = \sum_{j=0}^{(k,n-1)} 2(f_j h_{k+n-j} - h_j f_{k+n-j} + r_j r_{k+n-j}) \tag{3.3d}$$

$$q_{n;t_k} = \sum_{j=0}^{(k,n-1)} (h_j q_{k+n-j} - q_j h_{k+n-j} + e_j r_{k+n-j} - r_j e_{k+n-j}) \tag{3.3e}$$

$$r_{n;t_k} = \sum_{j=0}^{(k,n-1)} (r_j h_{k+n-j} - h_j r_{k+n-j} + f_j q_{k+n-j} - q_j f_{k+n-j}). \tag{3.3f}$$

We put $e_0 = f_0 = 0$, $q_0 = r_0 = 0$, which is compatible with (3.3a) but we leave h_0 as an undetermined constant. With $e=e_1, f=f_1, q=q_1, r=r_1, t_1=x$ we find that the functions e_k, f_k, q_k, r_k, h_k satisfy the following recursive relations:

$$e_{k+1}=(1/2h_0)[e_{k,x}+2(eh_k+qq_k)] \tag{3.4a}$$

$$f_{k+1}=(1/2h_0)[-f_{k,x}+2(fh_k+rr_k)] \tag{3.4b}$$

$$q_{k+1}=(1/h_0)(q_{k,x}+qh_k-er_k+re_k) \tag{3.4c}$$

$$r_{k+1}=(1/h_0)(-r_{k,x}+rh_k+fq_k-qf_k) \tag{3.4d}$$

$$h_k=D_x(ef_k-fe_k+rq_k+qr_k) \qquad D_x=\int_{-\infty}^{x}\mathrm{d}x. \tag{3.4e}$$

From (3.3) we find for an arbitrary k and $n=1$

$$e_{;t_k}=2h_0e_{k+1} \qquad f_{;t_k}=-2h_0f_{k+1} \tag{3.5a}$$

$$q_{;t_k}=h_0q_{k+1} \qquad r_{;t_k}=-h_0r_{k+1}. \tag{3.5b}$$

As the right-hand side of these equations is a polynomial expression in the variables e, f, q, r and their x derivatives, they represent a hierarchy of super-NLEE where for every fixed $k\geqslant 2$ we consider t_k as the time variable. We write down the first two non-trivial equations in the hierarchy.

(i) $k=2$:

$$e_{t_2}=(1/h_0)(\tfrac{1}{2}e_{xx}-e^2f+2req+2qq_x) \tag{3.6a}$$

$$f_{t_2}=(1/h_0)(-\tfrac{1}{2}f_{xx}+f^2e-2frq+2rr_x) \tag{3.6b}$$

$$q_{t_2}=(1/h_0)(q_{xx}-\tfrac{1}{2}qef+er_x+\tfrac{1}{2}re_x) \tag{3.6c}$$

$$r_{t_2}=(1/h_0)(-r_{xx}+\tfrac{1}{2}ref-fq_x-\tfrac{1}{2}qf_x). \tag{3.6d}$$

(ii) $k=3$:

$$e_{t_3}=(1/4h_0^2)(e_{xxx}-6ee_xf+12r_xeq+12req_x+12qq_{xx}) \tag{3.7a}$$

$$f_{t_3}=(1/4h_0^2)(f_{xxx}-6ff_xe-12frq_x+12fr_xq-12rr_{xx}) \tag{3.7b}$$

$$q_{t_3}=(1/h_0^2)(q_{xxx}-3/2q_xef-3/4qe_xf-3/4qef_x+3/2r_xe_x+3/4re_{xx}) \tag{3.7c}$$

$$r_{t_3}=(1/h_0^2)(r_{xxx}-3/2r_xef-3/4re_xf-3/4ref_x+3/2f_xq_x+3/4qf_{xx}). \tag{3.7d}$$

The functions h_k turn out to be closely related to the conserved quantities and to the Hamiltonians, as will be discussed in § 5. For later reference we list their values, corresponding to $k=0,1,2,3,4$:

$$h_0=c, \text{ an arbitrary constant} \tag{3.8a}$$

$$h_1=0 \tag{3.8b}$$

$$h_2=-(1/2h_0)ef+(1/h_0)rq \tag{3.8c}$$

$$h_3=(1/4h_0^2)(ef_x-fe_x)+(1/h_0^2)(rq_x-r_xq) \tag{3.8d}$$

$$h_4=(1/h_0^2)[-1/8h_0(ef_{xx}-e_xf_x+e_{xx}f)+1/h_0(rq_{xx}-r_xq_x+r_{xx}q) + \tfrac{3}{2}h_0h_2h_2+(3/2h_0)(rr_xe-qq_xf)]. \tag{3.8e}$$

3.2. $g = sl(2/1)$

We start with some general remarks on sl(m/n) which is defined as

$$\mathrm{sl}(m/n)=\left\{X=\begin{pmatrix}A & B\\ C & D\end{pmatrix};\ \mathrm{Str}\,X=\mathrm{Tr}\,A-\mathrm{Tr}\,D=0\right\} \tag{3.9}$$

where A is a $(m\times m)$ matrix, B a $(m\times n)$ matrix, C a $(n\times m)$ matrix, and D a $(n\times n)$ matrix. We assume $m\neq n$. Let E_{ab} be the matrix with entry 1 in the ath row and the bth column, and 0 elsewhere. The Cartan subalgebra is spanned by

$$H_a=E_{aa}-E_{a+1,a+1}\qquad a=1,\ldots,m-1,m+1,\ldots,m+n-1 \tag{3.10a}$$

$$H_m=E_{mm}+E_{m+1,m+1}. \tag{3.10b}$$

The even, respectively odd, roots are given by [4]:

$$\Delta_0=\{\varepsilon_a-\varepsilon_b,\ 1\leqslant a,b\leqslant m;\ \delta_c-\delta_d,\ 1\leqslant c,d\leqslant n\} \tag{3.11a}$$

$$\Delta_1=\{\pm(\varepsilon_a-\delta_c),\ 1\leqslant a\leqslant m,\ 1\leqslant c\leqslant n\} \tag{3.11b}$$

where ε_a, δ_c are linear functionals defined on the set of diagonal matrices $D=\{d=\mathrm{diag}(d_{1\,1},\ldots,d_{m+n,m+n})\}$ as follows:

$$\varepsilon_a(d)=d_{aa}\qquad \delta_c(d)=d_{m+c,m+c}\qquad a=1,\ldots,m\qquad c=1,\ldots,m. \tag{3.12}$$

A set of simple roots is given by

$$\Pi=\{\alpha_1,\ldots,\alpha_{m+n-1}\}=\{\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_m-\delta_1,\delta_1-\delta_2,\ldots,\delta_{n-1}-\delta_n\}. \tag{3.13}$$

From now on we consider the special case $g=\mathrm{sl}(2/1)$. A basis is given by

$$E_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\quad E_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\quad F_1=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\quad F_2=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$$

$$G_1=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\quad G_2=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\quad H_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad H_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{3.14}$$

i.e. four even elements E_1, F_1, H_1, H_2 and four odd elements E_2, F_2, G_1, G_2. The non-vanishing (anti)commutation relations are

$$\begin{aligned}
&[H_a,E_b]_-=K_{ab}E_b && [H_a,F_b]_-=-K_{ab}F_b && [E_a,F_b]_\mp=\delta_{ab}\\
&[H_a,G_b]_-=\Omega_{ab}G_b && [E_1,E_2]_-=G_1 && [E_1,G_2]_-=-F_2\\
&[E_2,G_2]_+=F_1 && [F_1,F_2]_-=-G_2 && [F_1,G_1]_-=E_2\\
&[F_2,G_1]_+=E_1 && [G_1,G_2]_+=H_1+H_2. &&
\end{aligned} \tag{3.15}$$

Here the Cartan matrix is $[K_{ab}]=\begin{pmatrix}2&-1\\-1&0\end{pmatrix}$ and $[\Omega_{ab}]=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$. Expand X_j in this basis:

$$X_j=h_j^aH_a+e_j^aE_a+f_j^aF_a+g_j^aG_a. \tag{3.16}$$

As in the previous example we use (2.9a) to put $e_0^a = f_0^a = g_0^a = 0$. Furthermore, we put $e_k^1 = e_k$, $f_k^1 = f_k$, $e_k^2 = \psi_k$, $f_k^2 = \varphi_k$, $g_k^1 = \chi_k$, $g_k^2 = \phi_k$ and $e_1 = e$, $f_1 = f$, $\psi_1 = \psi$, $\varphi_1 = \varphi$, $\chi_1 = \chi$, $\phi_1 = \phi$; $h_0^a \Omega_{a1} = -h_0^a \Omega_{a2} = \Omega_0$, $h_0^a K_{a1} = \lambda_1$, $h_0^a K_{a2} = \lambda_2$. Then the functions $e, f, \psi, \varphi, \chi, \phi$ satisfy the following t_k evolution equations:

$$
\begin{aligned}
&e_{t_k} = \lambda_1 e_{k+1} \qquad f_{t_k} = -\lambda_1 f_{k+1} \qquad \psi_{t_k} = \lambda_2 \psi_{k+1} \qquad \varphi_{t_k} = -\lambda_2 \varphi_{k+1} \\
&\chi_{t_k} = \Omega_0 \chi_{k+1} \qquad \phi_{t_k} = -\Omega_0 \phi_{k+1}
\end{aligned}
\tag{3.17}
$$

where e_{k+1}, f_{k+1}, ψ_{k+1}, φ_{k+1}, χ_{k+1}, ϕ_{k+1} are given by the coupled recursive relations:

$$e_{n+1} = (1/\lambda_1)(e_{n;x} + h_n^a e K_{a1} - \varphi \chi_n + \varphi_n \chi) \tag{3.18a}$$

$$f_{n+1} = (1/\lambda_1)(-f_{n;x} + h_n^a f K_{a1} + \psi \phi_n - \psi_n \phi) \tag{3.18b}$$

$$\psi_{n+1} = (1/\lambda_2)(\psi_{n;x} + h_n^a \psi K_{a2} - f\chi_n + f_n \chi) \tag{3.18c}$$

$$\varphi_{n+1} = (1/\lambda_2)(-\varphi_{n;x} + h_n^a \varphi K_{a2} + e_n \phi - e\phi_n) \tag{3.18d}$$

$$\chi_{n+1} = (1/\Omega_0)(\chi_{n;x} + h_n^a \chi \Omega_{a1} - e\psi_n + \psi e_n) \tag{3.18e}$$

$$\phi_{n+1} = (1/\Omega_0)(-\phi_{n;x} + h_n^a \phi \Omega_{a1} + \varphi f_n - f\varphi_n). \tag{3.18f}$$

For h_n^1 and h_n^2 we find

$$
\begin{aligned}
h_{n+1}^1 = -D_x[&(1/\lambda_1)(ef_{n;x} + fe_{n;x}) + (1/\Omega_0)(\chi\phi_{n;x} - \phi\chi_{n;x}) - (1/\lambda_1)(e\psi\phi_n + f\varphi\chi_n) \\
&- (1/\Omega_0)(\chi\varphi f_n + \phi\psi e_n) + (\lambda_2/\Omega_0\lambda_1)(e\psi_n\phi + f\varphi_n\chi)]
\end{aligned}
\tag{3.19a}
$$

$$
\begin{aligned}
h_{n+1}^2 = D_x[&(1/\lambda_2)(\varphi\psi_{n;x} - \psi\varphi_{n;x}) + (1/\Omega_0)(\phi\chi_{n;x} - \chi\phi_{n;x}) - (1/\lambda_2)(\psi e\phi_n + \varphi f\chi_n) \\
&- (1/\Omega_0)(\chi f\varphi_n + \phi e\psi_n) + (\lambda_1/\lambda_2\Lambda_0)(\psi e_n\phi + \varphi f_n\chi)].
\end{aligned}
\tag{3.19b}
$$

Equation (3.17) represents a recursive form of a hierarchy of super-NLEE. For $k = 1$ we find the trivial identities $e_{t_1} = e_x$, and the same for the other fields. For $k = 2$ the equations are as follows (where we have put $1/\lambda_1 = A$, $1/\lambda_2 = B$, $1/\Omega_0 = C$):

$$e_t = Ae_{xx} + 2Ae^2 f - C\chi\phi e + B\psi\varphi e - C\varphi\chi_x - B\varphi_x\chi \tag{3.20a}$$

$$f_t = -Af_{xx} + 2Af^2 e + C\chi\phi f - B\psi\varphi f + C\psi\phi_x + B\psi_x\phi \tag{3.20b}$$

$$\psi_t = B\psi_{xx} + Aef\psi + C\chi\phi\psi - Cf\chi_x - Af_x\chi \tag{3.20c}$$

$$\varphi_t = -B\varphi_{xx} - Aef\varphi - C\chi\phi\varphi - Ae_x\phi - Ce\phi_x \tag{3.20d}$$

$$\chi_t = C\chi_{xx} - Aef\chi + B\psi\varphi\chi - Be\psi_x + A\psi e_x \tag{3.20e}$$

$$\phi_t = -C\phi_{xx} + Aef\phi - B\psi\varphi\phi + A\varphi f_x - Bf\varphi_x. \tag{3.20f}$$

4. Reductions

Now we consider some special cases of the super-NLEE derived in § 3 by putting some restrictions on the fields. Some of the reduced equations will be identical to some previously considered in the literature. Others will be new.

4.1. g = b(0, 1)

We start with equations (3.6) and (3.7). If we put $f=\bar{e}, r=\bar{q}$ and $1/h_0=2\mathrm{i}$ these equations become respectively the super-NLS equation and the super-MKdV equation for one complex scalar field $e(t, x)$ and one complex anticommuting field $q(t, x)$ [7, 8]:

$$e_t=\mathrm{i}e_{xx}-2\mathrm{i}ee\bar{e}+4\mathrm{i}\bar{q}qe+4\mathrm{i}qq_x \tag{4.1a}$$

$$q_t=2\mathrm{i}q_{xx}-\mathrm{i}qe\bar{e}+2\mathrm{i}e\bar{q}_x+\mathrm{i}\bar{q}e_x \tag{4.1b}$$

$$e_t=-e_{xxx}+6ee_x\bar{e}-12\bar{q}_xeq-12\bar{q}eq_x-12qq_{xx} \tag{4.2a}$$

$$q_t=-4q_{xxx}+6q_xe\bar{e}+3qe_x\bar{e}+3qe\bar{e}_x+6\bar{q}_xe_x+3\bar{q}e_x. \tag{4.2b}$$

By setting $f=f_0=1$, $r=0$ and $1/h_0=2\mathrm{i}$ we find that (3.7) gives the super-KdV equation [7]:

$$e_t=-e_{xxx}+6ee_x-12qq_{xx} \tag{4.3a}$$

$$q_t=-4q_{xxx}+6q_xe+3qe_x. \tag{4.3b}$$

4.2. g = sl(2/1)

If we put $e=\bar{f}$, $\psi=\bar{\varphi}$, $\chi=\bar{\phi}$ and $1/\lambda_1=\mathrm{i}a$, $1/\lambda_2=\mathrm{i}b$, $1/\Omega_0=\mathrm{i}c$, where a, b, c are constants related by $c=ab/(a+b)$, the six equations (3.20) reduce to the following three coupled equations, for two complex anticommuting fields $\psi(x, t)$, $\chi(x, t)$ and one complex scalar field $e(x, t)$:

$$-\mathrm{i}e_t=ae_{xx}-2aee\bar{e}-c\chi\bar{\chi}e+b\psi\bar{\psi}e-c\bar{\psi}\chi_x-b\psi_x\chi \tag{4.4a}$$

$$-\mathrm{i}\psi_t=b\psi_{xx}+ae\bar{e}\psi+c\chi\bar{\chi}\psi-c\bar{e}\chi_x-a\bar{e}_x\chi \tag{4.4b}$$

$$-\mathrm{i}\chi_t=c\chi_{xx}-ae\bar{e}_\chi+b\psi\bar{\psi}\chi-be\psi_x+a\psi e_x. \tag{4.4c}$$

These equations generalise the super-NLS equations previously considered in the literature, as they contain cubic non-derivative terms in the spinor fields. The equations discussed in [6–8] are associated with super-Lie algebras, which only have one odd root and therefore only contain two spinor fields or a spinor field and its complex conjugate. The 'maximal' non-derivative spinor interaction is therefore of the form $\psi\chi$ or $\psi\bar{\psi}$ as $\psi^n=0$ for $n\geqslant 2$. The two odd roots for sl(2/1) therefore explain the presence of the cubic terms in (3.20) and (4.4).

5. Conservation laws and the Hamiltonian structure

As mentioned earlier, the functions h_n^a turn out to be intimately related to the conserved quantities. One therefore expects the Hamiltonians to be certain combinations of these functions. We have already written down, in (3.8), the explicit expressions for the first few functions h_n, in the case b(0, 1). These expressions are relatively easily derived because b(0, 1) only has two (simple) roots, one even and one odd.

Introduce the following Poisson bracket:

$$\{e(x), f(x')\} = h_0\delta(x-x') \qquad \{q(x), r(x')) = h_0\delta(x-x') \tag{5.1}$$

which vanishes for all other combinations of fields. We impose the following boundary condition on the fields $e(\pm\infty) = f(\pm\infty) = 0$, $q(\pm\infty)) = r(\pm\infty) = 0$. The integrals over $h_k(x)$ are in involution with respect to this Poisson bracket:

$$I_n = \int_{-\infty}^{+\infty} h_0 h_n(x)\,\mathrm{d}x \qquad \{I_n, I_m\} = 0 \tag{5.2}$$

and each one of them leads to evolution equations. In particular we find that (3.6) can be written as

$$e_t = \tfrac{1}{3}\{I_4, e\} \qquad f_t = \tfrac{1}{3}\{I_4, f\} \qquad q_t = \tfrac{1}{3}\{I_4, q\} \qquad r_t = \tfrac{1}{3}\{I_4, r\}. \tag{5.3}$$

To establish the Poisson commutativity we use the relation

$$h_{k+1}(x) = D_x h_{2;t_k}(x) \qquad k \in Z_+ \tag{5.4}$$

which is a special case of the equation

$$h_{n+1;t_k} - h_{k+1;t_n} = 0 \qquad k, n \in Z_+ \tag{5.5}$$

easily derived from (3.3b).

In the case $g = \mathrm{sl}(2/1)$ we find that for each simple root α^a, $a = 1, 2$, the functions h_k^a are not polynomial in e^a, f^a, g^a and x derivatives thereof. We find that the linear combinations

$$\begin{aligned} H_k &= \sum_{a,b=1}^{2} h_0^a K_{ab} h_k^b = \lambda_1 h_k^1 + \lambda_2 h_k^2 \\ &= -D_x(ef_{n;x} + fe_{n;x} + \chi\phi_{n;x} - \phi\chi_{n;x} - \varphi\psi_{n;x} + \psi\varphi_{n;x}) \end{aligned} \tag{5.6}$$

give polynomial expressions. For the first few values of k they are

$$H_0 = C, \text{ a constant} \tag{5.7a}$$

$$H_1 = 0 \tag{5.7b}$$

$$H_2 = -ef - \psi\varphi - \chi\phi \tag{5.7c}$$

$$H_3 = (1/\lambda_1)(ef_x - fe_x) + (1/\lambda_2)(\varphi\psi_x - \psi\varphi_x) + (1/\Omega_0)(\chi\phi_x - \phi\chi_x) \tag{5.7d}$$

$$\begin{aligned} H_4 = &-(1/\lambda_1)^2(ef_{xx} - e_x f_x + e_{xx} f) - (1/\Omega_0)^2(\chi\phi_{xx} - \chi_x\phi_x - \phi\chi_{xx}) \\ &+ (1/\lambda_2)^2(\varphi\psi_{xx} - \varphi_x\psi_x - \psi\varphi_{xx}) \\ &+ 3[(1/\lambda_1)^2 e^2 f^2 + (1/\lambda_1\Omega_0) ef\chi\phi + (1/\lambda_1\lambda_2) ef\varphi\psi + (1/\lambda_2\Omega_0)\chi\phi\varphi\psi] \\ &+ (1/\Omega_0\lambda_1\lambda_2)[(\lambda_2 + \Omega_0)(f_x\chi\varphi - e_x\psi\phi) + (\lambda_2 - \lambda_1)(e\psi\phi_x - f\chi_x\varphi) \\ &+ (\lambda_1 + \Omega_0)(e\psi_x\phi - f\chi\varphi_x)]. \end{aligned} \tag{5.7e}$$

Put the boundary condition on the fields $e(\pm\infty) = f(\pm\infty) = 0$, $\psi(\pm\infty) = \varphi(\pm\infty) = \chi(\pm\infty) = \phi(\pm\infty) = 0$. Let

$$K_n = \int_{-\infty}^{+\infty} H_n(x)\,\mathrm{d}x. \tag{5.8}$$

Then

$$\{K_n, K_m\} = 0 \tag{5.9}$$

where the Poisson bracket is defined as

$$\begin{aligned} &\{e(x), f(x')\} = \lambda_1 \delta(x - x') \qquad \{\varphi(x), \psi(x')) = \lambda_2 \delta(x - x') \\ &\{\chi(x), \phi(x')\} = \Omega_0 \delta(x - x'). \end{aligned} \tag{5.10}$$

For each simple root α^a, $a = 1, 2$, we find the relation

$$h^a_{n+1;t_k} - h^a_{k+1;t_n} = 0 \tag{5.11}$$

which can be used to establish the commutativity (5.9). Each K_n can be taken as a Hamiltonian for some evolution equation; in particular, we find that equations (3.20) can be written as

$$\begin{aligned} &e_t = \tfrac{1}{3}\{K_4, e\} \qquad f_t = \tfrac{1}{3}\{K_4, f\} \qquad \psi_t = \tfrac{1}{3}\{K_4, \psi\} \qquad \chi_t = \tfrac{1}{3}\{K_4, \chi\} \\ &\phi_t = \tfrac{1}{3}\{K_4, \phi\} \qquad \varphi_t = \tfrac{1}{3}\{K_4, \varphi\}. \end{aligned} \tag{5.12}$$

Further results can be obtained for other superloop algebras and will be published elsewhere.

6. Discussion

In this paper we have constructed supersystems of NLEE related to the super-Lie algebras b(0, 1) and sl(2/1). Every root vector of these algebras gives rise to a field, where even root vectors are mapped on commuting fields and odd root vectors on anticommuting fields. In the case of b(0, 1), which only has one even and one odd root, we find a system of NLEE in four fields, i.e. two scalar and two anticommuting fields. sl(2/1) has one even and two odd roots and therefore leads to NLEE in four anticommuting and two commuting scalar fields. The structure of these equations is considerably more complicated than the structure of the equations associated with b(0, 1), as is their infinite set of integrals.

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